2016/17 MATH2230B/C Complex Variables with Applications
Suggested Solution of Selected Problems in HW 3
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P.147 5 and P.170 3 will be graded

All the problems are from the textbook, Complex Variables and Application ($9th$ edition).

1 P.133

For the functions f and contours C in Exercise 8, use parametric representations for C or legs of C to evaluate

$$\int_C f(z)\,\mathrm{d}z.$$

8. $f(z)$ is the principal branch

$$z^{a-1} = \exp\big[(a-1)\mathrm{Log} z\big] \quad (|z|>0, -\pi < \mathrm{Arg} z < \pi)$$

of the power function z^{a-1}, where a is a nonzero real number and C is the positively oriented circle of radius R about the origin.

Solution. *The parametric representations for the contour is*

$$C = \{z = Re^{i\theta} : -\pi \le \theta \le \pi\}.$$

Then, one can obtain the integral as follows

$$\begin{aligned}\int_C f(z)\,\mathrm{d}z &= \int_{-\pi}^{\pi} e^{(a-1)(\ln R + i\theta)} \frac{d}{d\theta}(Re^{i\theta})\,\mathrm{d}\theta \\ &= iR(e^{(a-1)\ln R}) \int_{-\pi}^{\pi} e^{ia\theta}\,\mathrm{d}\theta \\ &= \frac{R^a}{a}\int_{-\pi}^{\pi} e^{ia\theta}\,\mathrm{d}ia\theta \\ &= \frac{R^a}{a}\big(e^{ia\pi} - e^{-ia\pi}\big) \\ &= \frac{2iR^a}{a}\sin a\pi.\end{aligned}$$

11. Let C denote the semicircular path

$$C = \{z : |z| = 2, \mathrm{Re}(z) \ge 0\}.$$

Evaluate the integral of the function $f(z) = \overline{z}$ along C using the parametric representation

(a) $z = 2e^{i\theta} \quad \Big(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\Big)$;

(b) $z = \sqrt{4-y^2} + iy \quad (-2 \le y \le 2)$.

Solution. *(a) The parametric representation of C is*

$$C = \left\{ z = 2e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}.$$

Hence, the integral can be calculated as follows:

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \int_C \overline{z}\,\mathrm{d}\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta} \frac{d}{d\theta}(2e^{i\theta})\,\mathrm{d}\theta \\ &= 4i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \\ &= 4\pi i. \end{aligned}$$

(b) The parametric representation of C is

$$C = \{z = \sqrt{4-y^2} + iy : -2 \leq y \leq 2\}.$$

Then, the integral can be calculated as follows:

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \int_C \overline{z}\,\mathrm{d}\theta \\ &= \int_{-2}^{2} (\sqrt{4-y^2} - iy) \frac{d}{dy}(\sqrt{4-y^2} + iy)\,\mathrm{d}y \\ &= \int_{-2}^{2} (\sqrt{4-y^2} - iy)\left(-\frac{y}{\sqrt{4-y^2}} + i \right) \mathrm{d}y \\ &= i \int_{-2}^{2} \left(\sqrt{4-y^2} + \frac{y^2}{\sqrt{4-y^2}} \right) \mathrm{d}y \\ &= 4i \int_{-2}^{2} \left(\frac{1}{\sqrt{4-y^2}} \right) \mathrm{d}y \\ &= 4\pi i. \end{aligned}$$

2 P.139

5. Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\mathrm{Log} z}{z^2}\,\mathrm{d}z \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

Proof. We take the principal branch of the logarithmic function and apply the following inequality to obtain the desired estimate:

$$\left| \int_{C_R} \frac{\mathrm{Log} z}{z^2}\,\mathrm{d}z \right| \leq \int_{C_R} \left| \frac{\mathrm{Log} z}{z^2} \right| \mathrm{d}z.$$

Further, we have, by the parametric representation of C

$$\begin{aligned}\left|\int_{C_R}\frac{\mathrm{Log}z}{z^2}\,\mathrm{d}z\right| &\leq \int_{-\pi}^{\pi}\frac{|\ln R+i\theta|}{R^2}|iRe^{i\theta}|\,\mathrm{d}\theta\\ &\leq \int_{-\pi}^{\pi}\frac{\ln R+\pi}{R}\\ &\leq 2\pi\left(\frac{\ln R+\pi}{R}\right).\end{aligned}$$

By the l'Hospital's rule, one can have

$$\lim_{R\to\infty}\frac{\ln R+\pi}{R}=\lim_{R\to\infty}\frac{1}{R}=0.$$

Hence, the integral tends to zero as $R\to\infty$. □

6. Let C_ρ denote a circle $|z|=\rho$ $(0<\rho<1)$ oriented in the counterclockwise direction and suppose that $f(z)$ is analytic in the disk $|z|\leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M independent of ρ such that

$$\left|\int_{C_\rho} z^{-1/2}f(z)\,\mathrm{d}z\right|\leq 2\pi M\sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

Proof. Since $f(z)$ is analytic in the disk $|z|\leq 1$, then it is continuous and bounded. There exists a positive constant $M>0$ such that

$$|f(z)|\leq M \quad \forall |z|\leq 1.$$

Then, the estimate can be obtained as follows

$$\begin{aligned}\left|\int_{C_\rho} z^{-1/2}f(z)\,\mathrm{d}z\right| &\leq \int_{-\pi}^{\pi}\frac{1}{\sqrt{\rho}}|e^{-i\theta/2}i\rho e^{i\theta}f(\rho e^{i\theta})|\,\mathrm{d}\theta\\ &\leq 2\pi M\sqrt{\rho}.\end{aligned}$$

Thus, the value of the integral approaches 0 as $\rho\to 0$. □

3 P.147

5. Show that

$$\int_{-1}^{1} z^i\,\mathrm{d}z=\frac{1+e^{-\pi}}{2}(1-i).$$

where the integrand denotes the principal branch

$$z^i=\exp(i\mathrm{Log}z)\quad (|z|>0.-\pi<\mathrm{Arg}z<\pi)$$

of z^i and where path of integration is any contour from $z=-1$ to $z=1$ that except for its end points lies above the real axis.

Proof. Note that $z^i = e^{i\text{Log}z} = e^{-\theta}e^{i\ln r}$ when the principal branch of the logarithmic function is selected. The parametric representation

$$C = \{z = x : -1 \le x \le 1\}$$

is given, then we can calculate the integral as follows

$$\begin{aligned}
\int_{-1}^{1} z^i \,\mathrm{d}z &= \int_{-1}^{0} z^i \,\mathrm{d}z + \int_{0}^{1} z^i \,\mathrm{d}z \\
&= \int_{-1}^{0} e^{-\pi}e^{i\ln(-x)} \,\mathrm{d}x + \int_{0}^{1} e^{0}e^{i\ln x} \,\mathrm{d}x \\
&= \int_{0}^{1} e^{-\pi}e^{i\ln x} \,\mathrm{d}x + \int_{0}^{1} e^{i\ln x} \,\mathrm{d}x \\
&= (1+e^{-\pi})\int_{0}^{1} e^{i\ln x} \,\mathrm{d}x \\
&= (1+e^{-\pi})\int_{0}^{1} (\cos(\ln x) + i\sin(\ln x)) \,\mathrm{d}x \\
&= (1+e^{-\pi})\int_{-\infty}^{0} e^{y}(\cos y + i\sin y) \,\mathrm{d}y \quad (y = \ln x) \\
&= (1+e^{-\pi})\int_{-\infty}^{0} e^{y(1+i)} \,\mathrm{d}y \\
&= \frac{1+e^{-\pi}}{1+i} \\
&= \frac{1+e^{-\pi}}{2}(1-i).
\end{aligned}$$

□

4 P.159

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. With the aid of the corollary in Sec. 53, point out why

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z$$

when

(a) $f(z) = \frac{1}{3z^2+1}$;

(b) $f(z) = \frac{z+2}{\sin(z/2)}$;

(a) $f(z) = \frac{z}{1-e^z}$.

Solution. *Try to explain that $f(z)$ is analytic inside the area between C_1 and C_2.*

(a) When $z \neq \pm\frac{\sqrt{3}}{3}i$, $f(z)$ is analytic.

(b) When $z \neq 2n\pi, n \in \mathbb{Z}$, $f(z)$ is analytic.

(c) When $z \neq 0$, $f(z)$ is analytic.

4. Use the following method to derive the integration formula

$$\int_0^\infty e^{-x^2} \cos 2bx \, \mathrm{d}x = \frac{\sqrt{\pi}}{2} e^{-b^2} \quad (b > 0).$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path $\{x = \pm a, y = 0 \text{ or } y = b\}$ can be written

$$2\int_0^a e^{-x^2} \, \mathrm{d}x - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx \, \mathrm{d}x$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2} \int_0^b e^{y^2 - i2ay} \, \mathrm{d}x - ie^{-a^2} \int_0^b e^{y^2 - i2ay} \, \mathrm{d}y.$$

Thus with the aid of the Cauchy-Coursat theorem, show that

$$\int_0^a e^{-x^2} \cos 2bx \, \mathrm{d}x = e^{-b^2} \int_0^a e^{-x^2} \, \mathrm{d}x + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay \, \mathrm{d}y.$$

(b) By accepting the fact that

$$\int_0^\infty e^{-x^2} \, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left| \int_0^b e^{y^2} \sin 2ay \, \mathrm{d}y \right| \leq \int_0^b e^{y^2} \, \mathrm{d}y$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

Proof. (a) The sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path can be written as

$$\int_{-a}^a e^{-x^2} \, \mathrm{d}x + \int_a^{-a} e^{-(x+ib)^2} \, \mathrm{d}x.$$

Simplify the above expression to obtain

$$\begin{aligned}
2\int_0^a e^{-x^2} \, \mathrm{d}x + e^{b^2} \int_a^{-a} e^{-x^2} e^{-i2bx} \, \mathrm{d}x &= 2\int_0^a e^{-x^2} \, \mathrm{d}x + e^{b^2} \int_a^{-a} e^{-x^2} (\cos 2bx - i \sin 2bx) \, \mathrm{d}x \\
&= 2\int_0^a e^{-x^2} \, \mathrm{d}x + e^{b^2} \int_a^{-a} e^{-x^2} \cos 2bx \, \mathrm{d}x \\
&= 2\int_0^a e^{-x^2} \, \mathrm{d}x - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx \, \mathrm{d}x.
\end{aligned}$$

Note that $e^{-x^2}\sin 2bx$ is odd function and $e^{-x^2}\cos 2bx$ is even function. Similarly, we can obtain the sum of integrals of e^{-z^2} along the vertical legs on the right and left

$$\int_0^b ie^{-(a+iy)^2}\,\mathrm{d}y + \int_b^0 ie^{-(-a+iy)^2}\,\mathrm{d}y.$$

Simplify the expression, we have:

$$ie^{-a^2}\int_0^b e^{y^2-i2ay}\,\mathrm{d}x - ie^{-a^2}\int_0^b e^{y^2-i2ay}\,\mathrm{d}y = 2e^{-a^2}\int_0^b e^{y^2}\sin 2ay.$$

By the Cauchy-Goursat theorem, the integral of e^{-z^2} along the rectangle is zero. It implies that

$$2\int_0^a e^{-x^2}\,\mathrm{d}x - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,\mathrm{d}x + 2e^{-a^2}\int_0^b e^{y^2}\sin 2ay = 0,$$

$$\int_0^a e^{-x^2}\cos 2bx\,\mathrm{d}x = e^{-b^2}\int_0^a e^{-x^2}\,\mathrm{d}x + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay.$$

(b) If we accept the fact that

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2},$$

and observe that

$$\left|\int_0^b e^{y^2}\sin 2ay\,\mathrm{d}y\right| \le \int_0^b e^{y^2}\,\mathrm{d}y \le be^{b^2}.$$

Then letting $a \to +\infty$, we have

$$\left|e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,\mathrm{d}y\right| \le be^{-a^2} \to 0.$$

Hence, we have the following formula

$$\int_0^\infty e^{-x^2}\cos 2bx\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}e^{-b^2} \quad (b>0).$$

□

5 P.170

2. Find the value of the integral of $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

   (a) $g(z) = \frac{1}{z^2+4}$;

   (b) $g(z) = \frac{1}{(z^2+4)^2}$.

Solution. *(a) Let* $f(z) = \frac{1}{z+2i}$*, then by Cauchy's integral formula, we have*

$$\int_C g(z)\,\mathrm{d}z = \int_C \frac{f(z)}{z-2i}\,\mathrm{d}z = 2\pi i \frac{1}{2i+2i} = \frac{\pi}{2}.$$

(b) Let $f(z) = \frac{1}{(z+2i)^2}$*, then by Cauchy's integral formula, we have*

$$\int_C g(z)\,\mathrm{d}z = \int_C \frac{f(z)}{(z-2i)^2}\,\mathrm{d}z = 2\pi i f'(2i) = \frac{\pi}{16},$$

where

$$f'(z) = -\frac{2}{(z+2i)^3}.$$

3. Let C be the circle $|z| = 3$ described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z}\,\mathrm{d}s \quad (|z| \neq 3)$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

Proof. Let $f(s) = 2s^2 - s - s$. By Cauchy's integral formula, we have

$$g(2) = \int_C \frac{f(s)}{s-2}\,\mathrm{d}s = 2\pi i f(2) = 8\pi i.$$

The value of $g(z)$ is 0 when $|z| > 3$ since the function $\frac{f(s)}{s-z}$ is analytic inside and on the contour C and by the Cauchy-Goursat theorem, the conclusion holds. □

4. Let C be any simple closed contour described in the positive sense in the z plane and write

$$g(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\,\mathrm{d}s.$$

Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside C.

Proof. By the Cauchy's integral formula, one can obtain

$$g(z) = \pi i f''(z) = 6\pi i z, \quad \forall z \text{ inside } C.$$

where $f(s) = s^3 + 2s$. For any z outside C, $g(z) = 0$ by the Cauchy-Goursat theorem. □